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14 Special Functions

Definition 1 (exponential function). The exponential function is defined by

exp:
$$\mathbb{R} \to \mathbb{R}$$

exp $(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$.

Theorem 2. The exponential function has the following properties.

- (*i*) exp is continuous on \mathbb{R} ;
- (*ii*) $\exp(x) > 0$ for all $x \in \mathbb{R}$;
- (iii) for all $x, y \in \mathbb{R}$: $\exp(x + y) = \exp(x) \cdot \exp(y)$;
- (iv) exp is indefinitely differentiable and exp' = exp;
- (v) exp is strictly increasing on \mathbb{R} ;
- (vi) $\lim_{x\to\infty} \exp(x) = 0$ and $\lim_{x\to\infty} \exp(x) = \infty$;
- (vii) exp: $\mathbb{R} \to (0, \infty)$ is bijective;
- (viii) $\exp(1) = e$ and $\exp(\frac{n}{m}) = e^{n/m}$ for $n, m \in \mathbb{Z}$ and $m \neq 0$.

Definition 3. For $x \in \mathbb{R}$ we define $e^x = \exp(x)$.

Definition 4 (natural logarithm). *The* (natural) logarithm ln : $(0, \infty) \rightarrow \mathbb{R}$ *is the inverse function of* exp.

Theorem 5. *The natural logarithm has the following properties.*

- (*i*) ln *is continuous on* $(0, \infty)$;
- (*ii*) $\ln(x) < 0$ for $x \in (0, 1)$, $\ln(1) = 0$ and $\ln(x) > 0$ for $x \in (1, \infty)$;
- (*iii*) for all $x, y \in (0, \infty)$: $\ln(xy) = \ln(x) + \ln(y)$;
- (iv) \ln is differentiable with $\ln'(x) = \frac{1}{x}$;
- (v) ln is strictly increasing on $(0, \infty)$;

- (vi) $\lim_{x\to 0} \ln(x) = -\infty$ and $\lim_{x\to\infty} \ln(x) = \infty$;
- (vii) $\ln: (0, \infty) \to \mathbb{R}$ is bijective.

Definition 6 (arbitrary powers). *For a*, $x \in \mathbb{R}$ *with a* > 0 *we define a*^{*x*} = $e^{x \ln(a)}$.

Theorem 7. The function $f(x) = a^x$ has the following properties.

- (i) $f : \mathbb{R} \to (0, \infty)$ is bijective;
- (*ii*) for all $x, y \in \mathbb{R}$: $f(x + y) = a^{x+y} = a^x a^y = f(x)f(y)$;
- (*iii*) for all $x, y \in \mathbb{R}$: $f(xy) = a^{xy} = (a^x)^y = f(x)^y$;
- (iv) f is differentiable with $f'(x) = f(x) \ln(x)$.

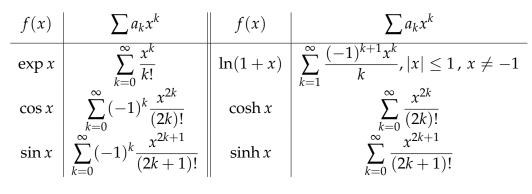
Definition 8 (arbitrary logarithm). *The inverse function of* $f(x) = a^x$ *is called* base *a* logarithm, *denoted by* \log_a .

Theorem 9. The base a logarithm can be computed as

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}.$$

Theorem 10. *The base a logarithm is differentiable with*

$$(\log_a(x))' = \frac{1}{x\ln(a)}.$$



Important functions